Riemannian Manifolds and Geodesics

With an aim to prove Hopf-Rinow Theorem

Soham Bakshi, B.Math 3rd Year

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Under guidance of Prof. Sugata Mondal

Riemannian Manifolds and Basic Constructs

Smooth manifold

Let M be a set. A chart on M is a pair (ϕ, U) where $U \subset M$ and ϕ is a bijection from U to an open subset $\phi(U) \subset \mathbb{R}^m$ of some Euclidean space. Two charts (ϕ_1, U_1) and (ϕ_2, U_2) are said to be smoothly compatible iff $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are both open in $\mathbb{R}^m$ and the transition map

$$\phi_{21} = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is a diffeomorphism. A smooth atlas on M is a collection A of charts on M any two of which are smoothly compatible and such that the sets U, as (ϕ, U) ranges over the elements of A, cover M (i.e. for every $p \in M$ there is a chart $(\phi, U) \in A$ with $p \in U$). A smooth manifold is a pair consisting of a set M and a maximal smooth atlas A on M.

Tangent Space

Let M be a smooth manifold. A smooth function $\alpha : (-\varepsilon, \varepsilon) \to M$ is called a (smooth) curve in M. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on M that are smooth at p. The tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbf{R}$ given by

$$\alpha'(0)f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}, \quad f \in \mathcal{D}$$

A tangent vector at p is the tangent vector at $t = 0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p is the tangent space (T_pM).

Tangent Space

Tangent space at a point $p \in M$ can be intuitively contains all possible directions passing ”tangentially” through it.

The tangent space at any point of a n dimensional smooth manifold is a n-dimensional vector space. If a chart $\varphi = \left(x^1, \ldots, x^n\right) : U \to \mathbb{R}^n$ is given with $p \in U$, then $\left\{ \left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)_p \right\}$ T_pM forms a basis.

Tangent Bundle

Tangent Bundle of M, denoted as TM is the union of all the tangent spaces.

$$TM = \bigcup_{p \in M} T_pM$$

TM is equipped with a smooth structure with dimension twice of M.

Vector Fields

A vector field X on a smooth manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into the tangent bundle TM. The field is smooth if the mapping $X : M \to TM$ is smooth.

$\mathcal{X}(M)$ denotes the set of all vector fields $X : M \to TM$.

Riemannian metric

Let M be a smooth m -manifold. A Riemannian metric on M is a collection of inner products

$$T_pM \times T_pM \to \mathbb{R} : (v, w) \mapsto g_p(v, w)$$

one for every $p \in M$, such that the map

$$M \to \mathbb{R} : p \mapsto g_p(X(p), Y(p))$$

is smooth for every pair of vector fields $X, Y \in \mathrm{Vect}(M)$.

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A smooth manifold equipped with a Riemannian metric is called a Riemannian manifold.

Affine Connection

An affine (or linear) connection ∇ on a smooth manifold M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

which is denoted by $(X, Y) \xrightarrow{\nabla} \nabla_X Y$ and which satisfies the following properties :

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$

in which $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

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in which $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

For any $V, W \in \mathcal{X}(M)$, $\nabla_W V$ is called the covariant derivative of V along W.

Vector Fields on Curves

A vector field along a curve $\gamma : I \to M$ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$. We let $\mathcal{X}(\gamma)$ denote the space of vector fields along γ.

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Let ∇ be a affine connection on M. For each curve $\gamma : I \to M$, ∇ determines a unique operator

$$D_t : \mathcal{X}(\gamma) \to \mathcal{X}(\gamma)$$

satisfying the following properties:

1. $D_t(aV + bW) = aD_tV + bD_tW \quad$ for $a, b \in \mathbf{R}$
2. $D_t(fV) = \dot{f}V + fD_tV \quad$ for $f \in C^\infty(I)$
3. If $V(t) = \widetilde{V}_{\gamma(t)}$ then $D_tV(t) = \nabla_{\dot{\gamma}(t)}\widetilde{V}$

Theorem
(Fundamental Lemma of Riemannian Geometry) Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ on M that is compatible with g and symmetric.

A linear connection ∇ is said to be compatible with g if it satisfies the following product rule for all vector fields X, Y, Z.

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

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The connection is called Levi-Civita connection.

Geodesics

Definition

Let M be a manifold with a linear connection ∇, and let γ be a curve in M.

The acceleration of γ is the vector field $D_t\dot{\gamma}$ along γ.

A curve γ is called a geodesic with respect to ∇ if its acceleration is zero: $D_t\dot{\gamma} \equiv 0$.

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Example: The geodesics on $\mathbb{R}^n$ are exactly the straight lines with constant speed parametrizations.

Existence and Uniqueness

Let M be a manifold with a linear connection. For any $p \in M$, any $V \in T_pM$, and any $t_0 \in \mathbf{R}$, there exist an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p, \dot{\gamma}(t_0) = V$. Any two such geodesics agree on their common domain.

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This determines a unique maximal geodesic γ_V for each $V \in TM$.

Exponential Map

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For each $p \in M$, the restricted exponential map $\exp_p$ is the restriction of $\exp$ to the set $\mathcal{E}_p := \mathcal{E} \cap T_pM$.

Properties

- (Rescaling) For any $V \in TM$ and $c, t \in \mathbb{R}$,

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  for all t such that either side is defined.
- The exponential map is smooth.

Normal Neighbourhoods

Fix $p \in M$, the restricted exponential map $\exp_p : \mathcal{E}_p \to M$, $\mathcal{E}_p$ open subset of T_pM.

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Theorem
For any $p \in M$, there is a neighborhood $\mathcal{V}$ of the origin in T_pM and a neighborhood $\mathcal{U}$ of p in M such that $\exp_p : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism.

(This follows from the inverse function theorem, once we show that $\left(\exp_p\right)_*$ is invertible at 0.)

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Any open neighbourhood $\mathcal{U}$ of $p \in M$ which is a diffeomorphic image of a open neighbourhood around $0 \in T_pM$ is called a normal neighbourhood of p.

Convex Sets

A set $S \subset M$ is convex if any two $p, q \in M$ can be connected by a geodesic.

In other words, S is a normal neighbourhood of all $p \in S$.

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Theorem
Every $p \in M$ is contained in a convex neighbourhood.

Length and Distance in Riemannian Manifolds

Curve Length

If $\gamma : [a, b] \to M$ is a curve segment, we define the length of γ to be

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| dt$$

The key feature of the length of a curve is that it is independent of parametrization.

Riemannian Distance

M is a connected Riemannian manifold.

For any pair of points $p, q \in M$, we define the Riemannian distance $d(p, q)$ to be the infimum of the lengths of all curves from p to q.

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Note: The topology induced by d metric is same as the given manifold topology. (Comparing riemmanian distance with euclidean distance in local coordinates)

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Theorem
Every Riemannian geodesic is locally minimizing.

Completeness and Hopf Rinow

Geodesically Complete

M is said to be geodesically complete iff for all $p \in M$, $\exp_p$ is defined for every $v \in T_pM$, i.e. all geodesics $\gamma(t)$ starting at p are defined for all $t \in \mathbb{R}$.

Examples: $\mathbb{R}^n$, $\mathbb{S}^n$, any compact manifold (why?).

Non-Example: $\mathbb{R}^2 \setminus \{(0,0)\}$

Theorem
Let M be a connected Riemannian manifold and $p \in M$. Then the following conditions are equivalent:

1. *M is geodesically complete.*
2. *$\exp_p$ is defined for every $v \in T_pM$.*
3. *Closed and bounded subsets of M are compact.*
4. *M is complete as a metric space.*

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3. *Closed and bounded subsets of M are compact.*
4. *M is complete as a metric space.*

Furthermore, these conditions also imply (but are not equivalent to): Any two points of M can be joined by a geodesic.

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Let $d(p,q) = r$ and let $B_\delta(p)$ be the normal ball at p and $S = S_\delta(p)$ is its boundary. Since $d(q,x)$ is continuous it attains a minimum on S which we will denote by x_0.

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Let $d(p,q) = r$ and let $B_\delta(p)$ be the normal ball at p and $S = S_\delta(p)$ is its boundary. Since $d(q,x)$ is continuous it attains a minimum on S which we will denote by x_0.

We find a unit vector $v \in T_pM$ such that $x_0 = \exp_p(\delta v)$. Now we define $\gamma(t) = \exp_p(tv)$.

We aim to show that $\gamma(r) = q$. We consider

$$A = \{s \in [0,r] | d(\gamma(s), q) = r - s\} \subset [0,r]$$

and show it's clopen in $[0,r]$ and thus $A = [0,r]$.

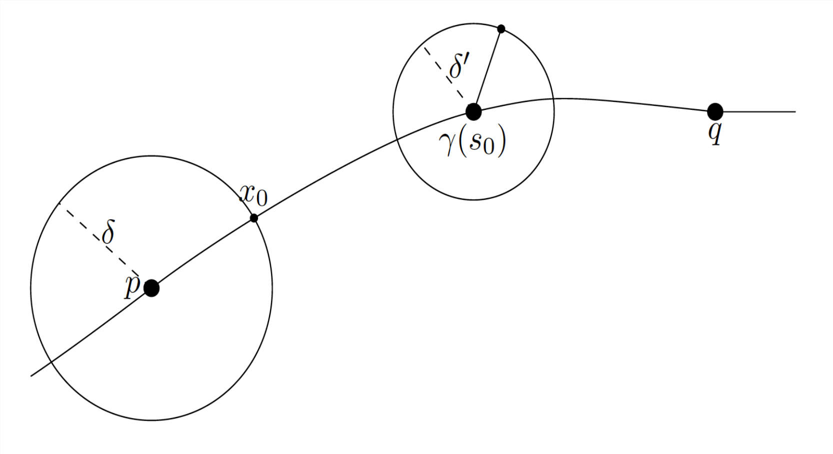

Figure 1: Hopf-Rinow

Cont.

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Take $B_{\delta'}(\gamma(s_0))$ be the normal ball with boundary S'. Let x_0' be the minimum of $d(x, q)$ on S'. By definition of the metric we have

$$r - s_0 = d\left(\gamma\left(s_0\right), q\right) = \delta' + \min_{x \in S'} d(x, q) = \delta' + d\left(x_0', q\right)$$

$$\implies d(x_0', q) = r - s_0 - \delta'.$$

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Left to show $\gamma(s_0 + \delta') = x_0'$.

Cont.

Using traingle inequality,

$$d\left(p, x_0'\right) \geq d(p, q) - d\left(q, x_0'\right) = r - \left(r - s_0 - \delta'\right) = s_0 + \delta'$$

The broken curve from p to x_0' via $\gamma\left(s_0\right)$ has length $s_0 + \delta'$ and hence needs to be a geodesic. So, it turns out to be

$$\gamma(s_0 + \delta') = x_0' \implies d(x_0', q) = r - s_0 - \delta' \implies s_0 + \delta' \in A$$

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Now, there is a ball $B_r(0) \subset T_pM$ such that $B_\varepsilon(p) \subset \exp_p\left(\overline{B_r(0)}\right)$.

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Now, there is a ball $B_r(0) \subset T_pM$ such that $B_\varepsilon(p) \subset \exp_p\left(\overline{B_r(0)}\right)$.

$\exp_p$ is continuous, the image on the right is compact and hence A as a closed subset of a compact set is compact itself.

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Let (x_n) be a Cauchy sequence in M. The set $\{x_n\}$ is bounded, so its closure is closed and bounded (so compact).

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Let (x_n) be a Cauchy sequence in M. The set $\{x_n\}$ is bounded, so its closure is closed and bounded (so compact).

Therefore (x_n) has a convergent subsequence and since (x_n) is Cauchy, it must converge to the limit of the subsequence.

$(4) \implies (1)$

Suppose M is complete as a metric space and there is some geodesic γ which cannot be continued for all time.

Let $t_0 = \sup\{t : \gamma(t) \text{ is defined }\}$. Then as $t \to t_0, \gamma(t)$ is a Cauchy sequence and so $\gamma(t_0)$ can be defined. (We use $|\gamma(s) - \gamma(t)| \leq |s - t|$)

We start at $\gamma(t_0)$ with initial velocity $\gamma'(t_0)$ and this shows we can continue the geodesic past t_0, a contradiction.

Thank You!